ordinate axis and the helix angle φ_1 is shown in fractions of $\pi/2$, i.e. $\bar{\varphi} = 2\varphi_1/\pi$.

The problem can also be solved using the elastic-plastic deformation equations of Sect.2. The only difference is that the deformation parameters (3.2) must be substituted into (1.12) instead of (2.9). The solution of the respective problem for an ideal elastic-plastic material and the deformation theory of plasticity when $\varphi_0 = \pi/2$, $R/a = 10^{-2}$ and $\tau_s/E = 7.2\cdot 10^{-3}$ are shown in Fig.2 by the solid lines. They are in good agreement with the results of the theory of limit equilibrium, beginning from the helix angles $\varphi < 0.8\pi/2$. The disagreement observed at $\varphi < 0.8\pi/2$ is explained by the fact that in the region of the $n = 0$ plane the approximating sphere lies inside the surface (2.9). The results are virtually indistinguishable when the accurate relations are used.

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AN APPROXIMATE METHOD OF OPTIMIZING THE SHAPE OF REINFORCEMENT RODS IN NON-UNIFORMLY AGING MATERIALS*

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The problem of optimizing the shape of a rod made of a non-uniformly aging viscoelastic material and reinforced by an elastic material is considered. Geometrical and integral constraints are imposed on the area of cross-section of the rod. The optimum shape is selected to minimize the maximum deflection of the rod in a fixed time interval. An approximate method of optimizing the shape is proposed and justified in the case of slight creep of the material. Results of numerical calculations are presented.

1. Statement of the problem of rod shape optimization. Consider the bending of a rod of length L made from non-uniformly aging viscoelastic material and reinforced by an elastic material. The 0ξ axis is directed along the axis of the rod in the undeformed state. We will denote by $I_0(\xi)$, I_a, $I(\xi)$ the moments of inertia of the cross-sections of the basic material, the reinforcing material, and the whole rod, respectively, and by $S(\xi)$ the rod cross-section at the point ξ. The arrangement of the reinforcement is specified, and is independent of the coordinate ξ. The rod moment of inertia $I(\xi)$ and the area of cross-section $S(\xi)$ are connected by the relation

$$I(\xi) = a_n S^n(\xi) \tag{1.1}$$

where n, a_n are given positive constants. The cross-sectional area of the rod is bounded

$$0 < S_1 \leqslant S(\xi) \leqslant S_2 < \infty \tag{1.2}$$

and the reinforcing material is completely covered by the viscoelastic material. The latter assumption is satisfied for example, when the reinforcement is in the region corresponding to the minimum possible area of cross-section that represents either a rectangle of constant thickness and varying width, or a rectangle of constant width and varying thickness, or a

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circle. The moduli E_0 of instantaneous elastic deformation of the basic material and E_a of the reinforcing material are constant, and the measure of creep of the viscoelastic material is defined /1/ by the formula $C_1(t,\tau)= \varphi_1(\tau)[1-\exp(-\gamma(t-\tau))]$, where φ_1 is a twice continuously differentiable, monotonically decreasing function that takes positive values, and $\gamma > 0$ are constant coefficients.

An external load is applied to the rod at the instant of time $t=0$ that acts during the time interval $[0, T]$. We denote by $Y(t, \xi)$ the rod deflection at the point ξ at the instant t, by $M(t, \xi)$ the bending moment $M(t, \xi) = M_1(t, \xi, Y(t, \xi))$, and by $\rho_1(\xi)$ the age of the basic material at the instant the external load is applied. The function ρ_1 is piecewise-continuous and bounded.

When the stress state is uniaxial, the strains e_0, e_a and stresses σ_0, σ_a of the basic and reinforcing materials are connected by the relations /2/

$$e_0(t, \xi) = \frac{\sigma_0(t,\xi)}{E_0} - \int_0^t \sigma_0(\tau, \xi)\frac{\partial}{\partial\tau} C_1(t+\rho_1(\xi), \tau+\rho_1(\xi))\,d\tau,\quad e_a(t,\xi) = \frac{\sigma_a(t,\xi)}{E_a}$$

It follows from the hypothesis of plane cross-sections and the conditions of continuity that the rod deflection satisfies the equation

$$\frac{\partial^2 Y(t,\xi)}{\partial\xi^2} - \varepsilon b\int_0^t \frac{\partial^2 Y(\tau,\xi)}{\partial\xi^2} K(t+\rho_1(\xi), \tau+\rho_1(\xi))\,d\tau = \tag{1.3}$$

$$\frac{b}{E_a I_a}\left[M(t,\xi) - \varepsilon\int_0^t M(\tau,\xi)K(t+\rho_1(\xi), \tau+\rho_1(\xi))\,d\tau\right]$$

where

$$K(t,\tau) = \partial C(t,\tau)/\partial\tau,\quad C(t,\tau) = C_1(t,\tau)/C_0 \tag{1.4}$$
$$b = b(\xi) = E_a I_a[E_a I_a + E_0 I_0(\xi)]^{-1},\quad C_0 = \lim\nolimits_{\tau\to\infty}\varphi_1(\tau)$$
$$\varphi(\tau) = \varphi_1(\tau)/C_0,\quad \varepsilon = E_0 C_0$$

The function b defines the degree of reinforcement of the basic material by the elastic one, the quantity C_0 defines the creep of the aged material, and the dimensionless parameter ε is the ratio between the elasticity and creep of the basic material. The numerical value of ε does not exceed unity for materials with strong creep properties (e.g., concrete), and is substantially less than unity for materials with low creep, Henceforth we will assume ε to be a small parameter.

The problem of optimizing the shape of a rod with a fixed volume V^0 consists of determining the function S_0 that satisfies (1.2) and minimizes the value of the rod maximum deflection

$$J = \sup\nolimits_{t,\xi}|Y(t,\xi)|,\ \int_0^L S_0(\xi)\,d\xi = V^0 \tag{1.5}$$
$$t\in[0, T],\quad \xi\in[0, L]$$

The function b aatisfies the equation

$$\beta_1 \leqslant b(\xi) \leqslant \beta_2,\quad \beta_{1,2} = E_a I_a[a_n E_0 S_{2,1}^n + (E_a - E_0)I_a]^{-1} \tag{1.6}$$

We take the function b as the new controlling function, since it is connected with S by the unique relation (1.1), (1.4).

2. Expansion of the solution of the optimization problem in series in powers of a small parameter. The control b^0 that satisfies (1.6) will be called the ε-optimal solution of the problem of optimizing the shape of a rod for a fixed volume, if a constant $c>0$ independent of ε exists such that $J(b^0) < J_0 + c\varepsilon$, where J_0 is the minimum value of the function (1.5) and the equation

$$\int_0^L\left[\frac{1}{b^0(\xi)} - \frac{E_a-E_0}{E_a}\right]^{1/n} d\xi = \left(\frac{a_n E_0}{E_a I_a}\right)^{1/n} V^0 = V_0 L \tag{2.1}$$

is satisfied.

Generally the precedure for determining the optimimum shape of the rod using the small parameter is as follows. We expand the function Y in a series in powers of ε, $Y(t,\xi) = Y_0 + \varepsilon Y_1 + \ldots + \varepsilon^j Y_j + \ldots$, substitute this expansion into (1.3), and equate the coefficients of like powers of ε. We obtain

$$\frac{\partial^2 Y_0}{\partial \xi^2} = \frac{b}{E_a I_a} M_1(t, \xi, Y_0), \quad \frac{\partial^2 Y_1}{\partial \xi^2} = \frac{b}{E_a I_a} \Big[\frac{\partial M_1(t, \xi, Y_0)}{\partial y} Y_1 - (1-b) \int_0^t M_1(\tau, \xi, Y_0) K(t + \rho_1(\xi), \tau + \rho_1(\xi))\, d\tau \Big] \tag{2.2}$$

$$\frac{\partial^2 Y_j}{\partial \xi^2} = \frac{b}{E_a I_a} \Big[\sum_{m=1}^{j} \frac{1}{m!} \frac{\partial^m M_1(t, \xi, Y_0)}{\partial y^m} \sum_{l_1+\ldots+l_m=j} Y_{l_1} \ldots Y_{l_m} - \sum_{m=1}^{j-1} \frac{1}{m!} \int_0^t \frac{\partial^m M_1(\tau, \xi, Y_0)}{\partial y^m} \sum_{l_1+\ldots+l_m=j-1} Y_{l_1} \ldots Y_{l_m} K(t + \rho_1(\xi), \tau + \rho_1(\xi))\, d\tau \Big] + b \int_0^t \frac{\partial^2 Y_{j-1}}{\partial \xi^2} K(t + \rho_1(\xi), \tau + \rho_1(\xi))\, d\tau, \quad j \geqslant 2$$

We denote by $Y(t, \xi, b)$, $Y_j(t, \xi, b)$ the solutions of (1.3) and (2.2) that correspond to control b, and by b_0, b_j^0 the controls that minimize the functionals $J(b) = \sup_{t,\xi} |Y(t, \xi, b)|$, $J_j(b) = \sup_{t,\xi} |Y_0(t, \xi, b) + \ldots + \varepsilon^j Y_j(t, \xi, b)|$ under conditions (1.6), (2.1). Let us assume that at fixed $j \geqslant 0$ a constant $c_1 \geqslant 0$ exists such that for any $t \in [0, T]$, $\xi \in [0, L]$, $\varepsilon \in (0, 1)$ that satisfy (1.6), (2.1) the inequality

$$|Y(t, \xi, b) - \sum_{k=0}^{j} \varepsilon^k Y_k(t, \xi, b)| \leqslant c_1 \varepsilon^{j+1} \tag{2.3}$$

is satisfied. Then the following estimates hold:

$$|J(b_j^0) - J_j(b_j^0)| \leqslant c_1 \varepsilon^{j+1}, \quad |J(b_0) - J_j(b_j^0)| \leqslant c_1 \varepsilon^{j+1} \tag{2.4}$$
$$|J(b_0) - J(b_j^0)| \leqslant 2c_1 \varepsilon^{j+1}$$

Relation (2.4) means that the function b_j^0 is the ε^{j+1}-optimal control in the problem of the rod shape optimization.

Remark. Let the external load and rod support conditions be independent of time. The functions Y_0, Y_1 have the form

$$Y_0(t, \xi) = Z_0(\xi), \quad Y_1(t, \xi) = (1 - e^{-\gamma t}) Z_1(\xi)$$

where Z_0, Z_1 satisfies the oridnary differential equations

$$\frac{d^2 Z_0}{d\xi^2} = \frac{b}{E_a I_a} M_1(\xi, Z_0) \tag{2.5}$$
$$\frac{d^2 Z_1}{d\xi^2} = \frac{b}{E_a I_a} \Big[\frac{\partial M_1(\xi, Z_0)}{\partial y} Z_1 + (1-b)\, \varphi_1(\rho_1(\xi)) M_1(\xi, Z_0) \Big]$$

If inequality (2.3) is satisfied, the problem of determining the optimal control of the integrodifferential equation (1.3) to within quantities of the order of ε^2 reduces to the problem of constructing the optimal control of ordinary differential equations (2.5).

We will further assume that the rod is subjected to a distributed transverse load of intensity $q_1 \geqslant 0$ and compressive force P. If the function q_1 is specified, the problem of shape optimization will be called the problem with full information. If, however, the function q_1 is a priori unknown, and only its equivalent force

$$Q = \int_0^L q_1(\xi)\, d\xi$$

is known, and the quantity $\sup_{t,\xi,q_1} |Y(t, \xi)|$ is to be minimized, the problem of rod shape optimization will be called the problem with insufficient information. To justify the estimate (2.4) it is then necessary to specify that the constant c_1 should be independent of q_1. The problem when there is no compressible force P was considered in /3/.

3. Optimization of the shape of a cantilever rod with incomplete information regarding the external load. Let one end of the rod be rigidly fixed and the other be free. The rod deflection is measured from the free end. Let us assume that the following inequality is satisfied:

$$PL^2 [a_n E_0 S_1^n + (E_a - E_0) I_a]^{-1} \leqslant 0{,}25\pi^2 \tag{3.1}$$

that ensures the stability of the respective elastic rod of any admissible form. Introducing the dimensionless variables

$$x = \xi/L, \quad q(x) = q_1(\xi)L/Q, \quad \rho(x) = \rho_1(\xi)$$
$$\beta(x) = b(\xi), \quad y(t, x) = PY(t, \xi)/(QL) \tag{3.2}$$

we write the equation for the deflection in the form

$$\frac{\partial^2 y(t,x)}{\partial x^2} + a\beta y(t,x) = -a\beta m\Big\{1 + \varepsilon\gamma(1-\beta)\varphi(\rho(x)) \times \int_0^t \exp\Big[-\gamma\Big(\tau + \varepsilon\beta\int_0^\tau \varphi(\xi+\rho(x))\,d\xi\Big)\Big]d\tau\Big\} - \varepsilon a\gamma\beta(1-\beta)\int_0^t y(\tau,x)\Big\{\varphi(\tau+\rho(x)) - \frac{\partial}{\partial\tau}\Big(\varphi(\tau+\rho(x)) \times \exp\Big[\gamma\Big(\tau+\varepsilon\beta\int_0^\tau \varphi(\xi+\rho(x))\,d\xi\Big)\Big]\Big) \times \int_\tau^t \exp\Big[-\gamma\Big(\xi+\varepsilon\beta\int_0^\xi \varphi(\eta+\rho(x))\,d\eta\Big)\Big]d\xi\Big\}\,d\tau, \quad a = PL^2/(E_aI_a) \tag{3.3}$$

$$y(t,1) = \partial y(t,0)/\partial x = 0, \quad m(x) = \int_0^1 q(\xi)(\xi - x)\,d\xi$$

where a is a dimensionless parameter.

For any admissible functions β the rod deflection reaches its maximum value at $t = T$, $x = 0$, $q(x) = \delta(x)$, where δ is the delta function, i.e. when the transverse load represent the concentrated load Q applied to the free end of the rod.

We denote by B the set of functions β measurable on the segment $[0, 1]$ satisfying (1.6), and by B_0 the set of functions $\beta_0 \in B_0$ that satisfy (2.1).

The problem of optimizing the shape of the rod consists of determining the functions $\beta_0 \in B_0$ that minimize the functional $J(\beta) = y(T, 0)$, where the function y satisfies (3.3) when $m(x) = m_0(x) = 1 - x$. We set $z_0(x) = y(0, x)$, $z_1(t, x) = e^{\gamma t}\partial y(t, x)/\partial x$. The functions z_0, z_1 satisfy the equations

$$\frac{d^2 z_0}{dx^2} + a\beta z_0 = -a\beta m(x) \tag{3.4}$$

$$\frac{\partial^2 z_1(t,x)}{\partial x^2} + a\beta z_1(t,x) = -a\varepsilon\gamma\beta(1-\beta)\Big[\int_0^t z_1(\tau,x)\varphi(\tau+\rho(x)) \times \exp\Big(\varepsilon\gamma\beta\int_0^\tau \varphi(\xi+\rho(x))\,d\xi\Big)d\tau + (z_0(x) + m(x))\varphi(\rho(x))\Big] \times \exp\Big(-\varepsilon\gamma\beta\int_0^t \varphi(\tau+\rho(x))\,d\tau\Big)$$

$$z_0(1) = \frac{dz_0(0)}{dx} = 0, \quad z_1(t,1) = \frac{\partial z_1(t,0)}{\partial x} = 0$$

Using the method of Lagrange multipliers we form the expression

$$J^{(\lambda)}(\beta) = z_0(0) + \int_0^T z_1(t,0)\exp(-\gamma t)\,dt + \lambda\Big(\int_0^1 \Big[\frac{1}{\beta(x)} - \frac{E_a - E_0}{E_a}\Big]^{1/n}dx - V_0\Big)$$

Let us consider the problem of determining the optimal control $\beta_0^{(\lambda)} \in B$ that minimizes the functional $J^{(\lambda)}$ on the set B. The optimal control in the initial problem has the form $\beta_0 = \beta_0^{(\lambda_0)}$, where λ_0 is found from the condition

$$\int_0^1 \Big[\frac{1}{\beta_0^{(\lambda_0)}(x)} - \frac{E_a - E_0}{E_a}\Big]^{1/n}dx = V_0$$

We denote by ψ the solution of the conjugate equation

$$\frac{\partial^2\psi(t,x)}{\partial x^2}+a\beta\psi(t,x)=-\varepsilon a\gamma\beta(1-\beta)\varphi(t+\rho(x))\times$$
$$\int_t^T\psi(\tau,x)\exp\left[-\gamma\int_t^\tau(1+\varepsilon\beta\varphi(\xi+\rho(x)))\,d\xi\right]d\tau;\quad \psi(t,1)=0,\quad \partial\psi(t,0)/\partial x=-1$$

and by F, G the functions of the form

$$F(t,x,\beta)=z_1(t,x)+\varepsilon\gamma(1-2\beta)\Big[\int_0^t z_1(\tau,x)\varphi(\tau+\rho(x))\times$$
$$\exp\left(\varepsilon\gamma\beta\int_0^\tau\varphi(\xi+\rho(x))\,d\xi\right)d\tau+(z_0(x)+m(x))\varphi(\rho(x))\times$$
$$\exp\left(-\varepsilon\gamma\beta\int_0^t\varphi(\xi+\rho(x))\,d\xi\right)\Big]-\varepsilon^2\gamma^2\beta(1-\beta)\times\Big[\int_0^t z_1(\tau,x)\varphi(\tau+\rho(x))\int_\tau^t\varphi(\xi+\rho(x))\,d\xi\times$$
$$\exp\left(-\varepsilon\gamma\beta\int_\tau^t\varphi(\xi+\rho(x))\,d\xi\right)d\tau+(z_0(x)+m(x))\times$$
$$\varphi(\rho(x))\int_0^t\varphi(\xi+\rho(x))\,d\xi\exp\left(-\varepsilon\gamma\beta\int_0^t\varphi(\xi+\rho(x))\,d\xi\right)\Big]$$
$$G(x,\beta)=a\Big[\psi(0,x)(z_0(x)+m(x))+\int_0^T\psi(t,x)F(t,x,\beta)\exp(-\gamma t)\,dt\Big]$$

Let $\beta_0(\lambda,x)$ be the solution of the algebraic equation

$$nG(x,\beta)\beta^{1+1/n}\left(1-\frac{E_a-E_0}{E_a}\beta\right)^{1-1/n}=\lambda$$

According to the necessary optimality condition /4/, the function $\beta_0^{(\lambda)}$ is defined by the formula

$$\beta_0^{(\lambda)}=\begin{cases}\beta_1, & \beta_0(\lambda,x)<\beta_1\\ \beta_2, & \beta_0(\lambda,x)>\beta_2\\ \beta_0(\lambda,x), & \beta_1\leqslant\beta_0(\lambda,x)\leqslant\beta_2\end{cases}$$

In the case of full information about the external load the above formulas remain valid, if by $m(x)$ we mean the dimensionless bending moment of the external load.

To investigate the effect of the basic material age on the optimum shape of the rod, a numerical solution was obtained for the problem of optimizing a rod of rectangular cross-section of thickness h and constant width d when there is no reinforcing material. The selected

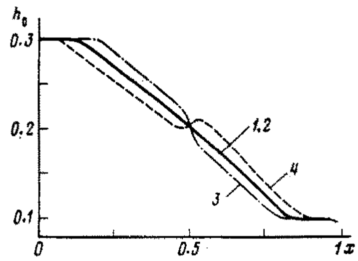

Fig.1

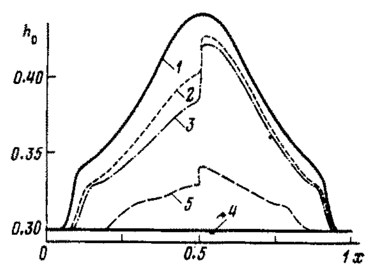

Fig.2

parameters of the problem were: $L=4$ m, $d=0.3$ m, $h_1=0.1$ m, $h_2=0.3$ m, $\varphi_1(\tau)=A_0+A_1/\tau$, $A_0=0.238\cdot10^{-4}$ MПa^{-1}, $A_1=1.85\cdot10^{-4}$ MПa$^{-1}\cdot$ day, $E_0=2.0\cdot10^4$ MПa, $\gamma=0.04$ day^{-1} $T=50$ day, $V^0=0.24$ m^3. The rod is subjected to a uniformly distributed transverse load and compressive force $P=2.5\cdot10^4$ N. As the test functions ρ_1 we used the following:

$$1)\ \rho_1(\xi)=2\text{ day};\quad 2)\ \rho_1(\xi)=15\text{ day};\quad 3)\ \rho_1(\xi)=\begin{cases}2\text{ day}, 0\leqslant\xi\leqslant2\text{ m}\\ 15\text{ day}, 2<\xi\leqslant4\text{ m}\end{cases};$$
$$4)\ \rho_1(\xi)=\begin{cases}15\text{ day}, 0\leqslant\xi\leqslant2\text{ m}\\ 2\text{ day}, 2<\xi\leqslant4\text{ m}\end{cases}$$

The optimal thickness distribution of the rod is shown in Fig.1, where curves *1–4* correspond to the above test functions. The calculations show that the aging of the whole rod material virtually does not affect the optimal thickness distribution, while a change in the age of one the parts of the rod results in a redistribution of the material and an increase in the younger part of the rod at the expense of the older part.

The solution of the equations that determine the optimum shape of the cantilever rod is a fairly complicated problem, particularly when a reinforcing material is present. It is more convenient to use the method of Sect.2 of expanding the solution of the optimization problem in series in powers of the small parameter, and determining the ε^2-optimal shape of the rod. When condition (3.1) is satisfied for any function $\beta \in B$, formula (2.3) holds for constant c_1 independent of q,t,x. Hence the ε^2-optimal control in the problem of optimizing the shape of the cantilever rod provides the minimum of the functional

$$J_1(\beta) = y_0(0) + \varepsilon\,[1 - \exp(-\gamma T)]\, y_1(0)$$

where y_0, y_1 is the solution of the set of equations

$$\frac{d^2y_0}{dx^2} + a\beta y_0 = -a\beta m(x), \quad y_0(1) = \frac{dy_0(0)}{dx} = 0 \tag{3.5}$$

$$\frac{d^2y_1}{dx^2} + a\beta y_1 = -a\beta(1-\beta)\varphi(\rho(x))[y_0 + m(x)], \; y_1(1) = \frac{dy_1(0)}{dx} = 0$$

To solve the problem of the optimal control of the set of equations (3.5) we can use the Pontriagin maximum principle /5/.

Above we succeeded in obtaining, in explicit form, the solution of the problem of optimizing the rod shape; hence it is possible to show beforehand that for chosen conditions of fixing the ends, at what point the rod deflection reaches its maximum value. For other cases of end support it is difficult to show in advance the point of maximum deflection, and the optimization problem is more complicated. It is then more convenient to solve not the initial problem but the converse problem of optimizing the rod shape for a given maximum value of the deflection Y°. It consists in determining the function S_0 that satisfies (1.2) and minimizes the rod volume with the condition

$$\sup_{t,\xi} |Y(t,\xi)| \leqslant Y^\circ \tag{3.6}$$

The solutions of the original and converse optimization problems are identical in the following sense. Let V° be the minimum rod volume for which the functional (1.5) does not exceed Y°. Then V° is the minimum value of the quality functional in the converse problem, and the optimal shapes of the rod in the original and converse problems are identical. Conversely, let Y° be the minimum value of the right-hand side of (3.6) for which the minimum value of the rod volume in the converse problem is V°. Then, Y° is the minimum quality functional in the original problem, and the optimal rod shapes in the original and the converse problems are identical.

The control $\beta^\circ \in B$ that satisfies (3.6) will be called the ε-optimum solution of the reciprocal problem, if a constant $c \geqslant 0$ exists independent of ε such that the maximum deflection that correspond to this control does not exceed $Y^\circ + c\varepsilon$ and the rod volume does not exceed $V(Y^\circ) + c\varepsilon$, where $V(Y^\circ)$ is the minimum volume in the converse problem.

4. Optimization of the shape of a hinged rod with full information about the external load. Let both ends of the rod be hinged and let the rod be subjected to a distributed load of intensity $q_1 \geqslant 0$ and a compressive force P. The converse problem of rod shape optimization consists in determining the function $\beta_0 \in B$ that minimizes the functional

$$V(\beta) = \int_0^1 \left[\frac{1}{\beta(x)} - \frac{E_a - E_0}{E_a}\right]^{1/n} dx \tag{4.1}$$

with the condition

$$\sup_{t,x} |y(t,x)| \leqslant y^\circ = PY^\circ/(QL) \tag{4.2}$$

When inequality (3.1) is satisfied for any $t \in [0, T]$, $x \in [0, 1]$, $\beta \in B$, the rod deflection is non-negative and reaches its maximum value at $t = T$. Hence, it is possible to substitute for (4.2) the expression

$$\sup_x \left[z_0(x) + \int_0^T z_1(t,x)\exp(-\gamma t)\,dt\right] \leqslant y^\circ \tag{4.3}$$

We will solve the optimal control problem (3.4), (4.1), and (4.3) using the penalty method /6/. We fix the sequence of positive numbers $\{\mu_m\}$, $\lim_{m\to\infty} \mu_m = \infty$, and consider the sequence of minimization problems on the set B of functionals

$$V_m(\beta) = V(\beta) + \mu_m \int_0^1 \left[\max\left(z_0(x) + \int_0^T z_1(t,x)e^{-\gamma t}\,dt - y^\circ, 0\right)\right]^2 dx \tag{4.4}$$

We denote by β_m° the optimal control in problem (3.4), (4.4), and by ψ the solution of the conjugate equation

$$\frac{\partial^2\psi(t,x)}{\partial x^2} + a\beta\psi(t,x) = -\varepsilon a\gamma\beta(1-\beta)\varphi(t+\rho(x))\int_t^T \psi(\tau,x)\times$$
$$\exp\left[-\gamma\int_t^\tau (1+\varepsilon\beta\varphi(\xi+\rho(x)))\,d\xi\right]d\tau - 2\mu_m \max\left[z_0(x)+\right.$$
$$\left.\int_0^T z_1(t,x)\exp(-\gamma t)\,dt - y^\circ, 0\right], \quad \psi(t,0)=0, \quad \psi(t,1)=0$$

Using the necessary condition of optimality /4/, we obtain that for any m the function β_m° is determined by the formula

$$\beta_m^\circ = \begin{cases} \beta_1, & \beta_0(1,x) < \beta_1 \\ \beta_2, & \beta_0(1,x) > \beta_2 \\ \beta_0(1,x), & \beta_1 \leqslant \beta_0(1,x) \leqslant \beta_2 \end{cases}$$

It can be shown that for any m the control $\beta_m^\bullet$ is the $\mu_m^{-1/4}$-optimum control in the problem of optimizing the rod shape.

To investigate the effect of basic material age and the magnitude of the maximum admissible deflection on the optimum shape of the rod the problem of shape optimation was solved numerically for a rod of rectangular cross-section of constant width and varying thickness without reinforcing material. The following parameters of the problem were selected: $L = 10\,\mathrm{m}$, $d = 0.5\,\mathrm{m}$, $h_1 = 0.3\,\mathrm{m}$, $h_2 = 0.5\,\mathrm{m}$, $T = 25$ days. The rod is subjected to a uniformly distributed transverse load of intensity $q_1 = 10^4$ N and a force $P = 2.5\cdot 10^4$ N. We chose the following as test functions ρ_1 :

$$1)\ \rho_1(\xi) = 3\,\mathrm{day}; \quad 2)\ \rho_1(\xi) = \begin{cases} 7\,\mathrm{day}, 0 \leqslant \xi \leqslant 5\ \mathrm{m} \\ 3\,\mathrm{day}, 5 < \xi \leqslant 10\ \mathrm{m} \end{cases};$$
$$3)\ \rho_1(\xi) = \begin{cases} 20\,\mathrm{day}, 0 \leqslant \xi \leqslant 5\ \mathrm{m} \\ 3\,\mathrm{day}, 5 < \xi \leqslant 10\ \mathrm{m} \end{cases}$$

The optimal distribution of the rod thickness for $Y^0 = 4.7\cdot 10^{-2}\,\mathrm{m}$ is represented in Fig.2, where curves *1–3* correspond to the above test functions. Curves *4* and *5* correspond to $Y^0 \geqslant 12.3\cdot 10^{-2}\,\mathrm{m}$, and $Y^0 = 8.7\cdot 10^{-2}\,\mathrm{m}$, when the age of the basic material is defined by test function 2). The numerical analysis shows that as the age of one of the parts of the rod material increases its volume decreases, and a partial redistribution of material from the region of older to that of younger material occurs. When the maximum admissible deflection Y^0 is reduced, the rod volume increases, and the rod seems to swell, retaining its general shape.

5. Approximate solution of the problem of rod shape optimization with complete information about the external load. The algorithm for solving the converse problem of rod shape optimization proposed in Sect.4 involves solving a system of non-linear integro-differential equations, and depends on the choice of the sequence of penalty coefficients $\{\mu_m\}$.

In the case of a small parameter the algorithm can be improved by coordinating the choice of the penalty coefficients and the value of the small parameter. The initial problem then reduces to the problem of the optimal control of a system of the form (2.2) without constraints on the phase coordinates. The latter problem is substantially simpler than the input one, and the fixed parameter can be determined using well-known numerical methods.

To give a specific example, we will apply the proposed algorithm to a hinged rod.

We will represent the magnitude of the deflection in the form of a series in powers of

$$y(t,x) = y_0 + \varepsilon y_1 + \ldots + \varepsilon^j y_j + \ldots \tag{5.1}$$

Substituting expansion (1.5) into Eq.(3.3) and equating the coefficients of like powers of the small parameter, we obtain a set of equations similar to (2.2). We introduce the following notation:

$$\eta_j(x) = y_0(T,x) + \varepsilon y_1(T,x) + \ldots + \varepsilon^j y_j(T,x)$$

Besides the integral parameter j we introduce the parameter l, $0 < l < j$, and consider the problem of minimizing on the set B the functionals

$$V_l(\beta) = V(\beta) + \varepsilon^{-l}\int_0^1 \max[y(T,x) - y^\circ, 0]\,dx \tag{5.2}$$
$$V_{j,l}(\beta) = V(\beta) + \varepsilon^{-l}\int_0^1 \max[\eta_j(x) - y^\circ, 0]\,dx$$

We denote by β_l°, $\beta_{j,l}^\circ$ the controls which minimize the functionals (5.2), by $\eta_{j,l}(x)$ the value of the function $\eta_j(x)$ that corresponds to the control $\beta_{j,l}^\circ$, and by $\eta_{j,l}^\circ$ the maximum value of the function $\eta_{j,l}$ on the segment $[0, 1]$. It can be shown that a constant $c_2 > 0$ exists independent of ε such that the inequalities

$$V(\beta_{j,l}^\circ) \leqslant V(\beta_0) + c_2\varepsilon^{j-l+1} \tag{5.3}$$

$$\eta_{j,l}^\circ \leqslant y^\circ + c_2\varepsilon^{2l/3}, \quad \sup_x y(T, x, \beta_{j,l}^\circ) \leqslant y^\circ + c_2\varepsilon^{2l/3} \tag{5.4}$$

are satisfied.

We will select the parameter l from the condition that the degree of error with respect to functional (5.3) and with respect to the maximum value of the deflection (5.4) are equal. Then when $l = 0,6(j+1)$, the optimal control of the set of equations (2.2) that minimizes the quality criterion $V_{j,l}(\beta)$ determines the $\varepsilon^{0,4(j+1)}$-optimal control in the converse problem of rod shape optimization.

In particular, when $j = 2$, the optimal control of the system of ordinary differential equations

$$\frac{d^2y_0}{dx^2} + a\beta y_0 = -a\beta m(x)$$

$$\frac{d^2y_1}{dx^2} + a\beta y_1 = -a\beta(1-\beta)\varphi(\rho(x))(y_0 + m(x)).$$

$$\frac{d^2y_2}{dx^2} + a\beta y_2 = -a\gamma\beta(1-\beta)\left\{y_1 - \beta\varphi(\rho(x))(y_0 + m(x)) \times \int_0^T \varphi(t+\rho(x))[\exp(-\gamma t) - \exp(-\gamma T)]\,dt\right\}$$

is the zero boundary condition that minimizes the functional

$$V(\beta) + \varepsilon^{-4/3}\int_0^1 \max[\eta_2(x) - y^\circ, 0]\,dx$$

$$\eta_2(x) = y_0(x) + \varepsilon(1 - \exp(-\gamma T))\,y_1(x) + \varepsilon^2 y_2(x)$$

The proposed algorithm of optimal shape determination can be applied to rods with other forms of support. It is then only necessary to investigate the supplementary conditions that guarantee that inequality (2.3) is satisfied.

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RATIONAL SCHEMES FOR REINFORCING LAMINAR PLATES FROM COMPOSITE MATERIALS*

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New problems for optimizing the internal structure of plates from a laminar composite for a number of local and integral functionals are considered. A model of a laminar-fibrous composite plate is described. Prior to optimization, the plate is a packet of monolayers homogeneous over the thickness. The monolayers are formed by periodic unidirectional stacking of reinforcing fibres in an elastic matrix. To determine the effective elastic properties of the monolayers, a homogenized model of the composite material is used. The concentration of reinforcing fibres or the angles of orientation of the axes of material anisotropy in a given number of

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